Circular Motion :-

When a particle moves along the circumference of a circle, its motion is called circular motion.

Examples :-

1] Motion of the tip of the blades of fan.

2] motion of swing (chair) of merry go round.

Types of circular motion,

There are two types of circular motion,

1] Uniform circular motion (U.C.M.)

2] Non uniform circular motion (Non U.C.M)

1] Uniform circular motion :-

The motion of a particle along the circumference of a circle with a constant linear speed, angular speed & angular velocity is called uniform circular motion.

Examples :-

1] Motion of the tip of hands of clock.

2] motion of the Earth around the sun.

Non-uniform circular motion :-

The motion of a particle along the circumference of a circle with a variable linear speed, linear velocity angular velocity is called Non Uniform circular motion.

Example :-

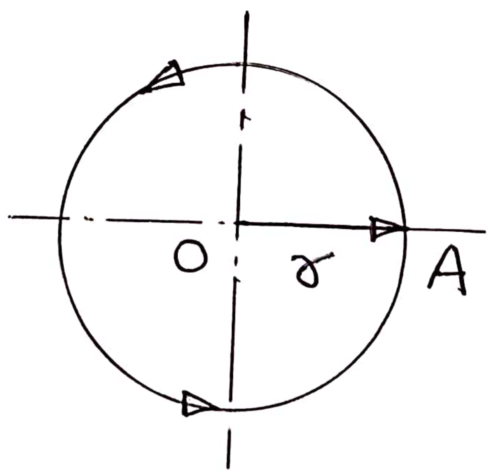
1] motion in a verticle circle.

Radius vector :-

The vector drawn from the centre of the circle to the position of the particle performing circular motion is called the radius vector.

S.I. Unit :- metre (m)

Dimensions :- $[M^0L^1T^0]$



1) Angular Displacement :-

The angle described by radius vector at the centre of the circle.

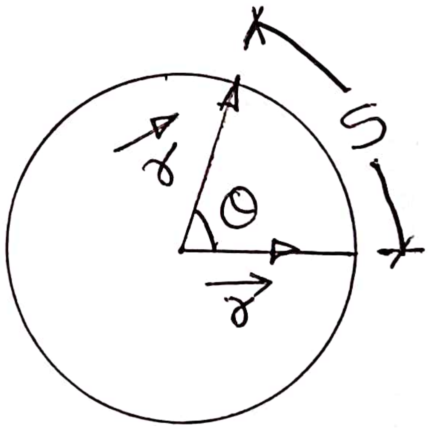
Or

When the particle moves from one point to other point on the circumference of the circle is called angular displacement.

It is denoted by 'θ'

mathematically it is written as,

$$\boxed{\theta = \frac{\text{arc length}}{\text{Radius}}}$$



$$\boxed{\theta = \frac{s}{r}}$$

S.I. unit is :- "radian"

Dimensions :- $[M^0 L^0 T^0]$

Angular velocity :-

The time rate of change of angular displacement of a particle is called angular velocity.

it is denoted by ω (omega)

S.I. unit is rad/s

Dimensions :- $[M^0 L^0 T^{-1}]$

mathematically it is written as,

$$\boxed{\omega = \frac{\theta}{t}}$$

Angular speed :- (ω)

The angle described by the particle per unit time is called angular speed.

It is denoted by ω

S.I. unit is :- rad/s

Dimensions :- $[M^0 L^0 T^{-1}]$

Angular acceleration :-

- The time rate of change of angular velocity of a particle is called angular acceleration.

it is denoted by 'α'

S.I. Unit is :- rad/s^2

Dimensions :- $[M^0 L^0 T^{-2}]$

mathematically it is written as,

$$\alpha = \frac{\text{change of angular velocity}}{\text{time}}$$

$$\boxed{\alpha = \frac{\omega_2 - \omega_1}{t}}$$

Period :- (T)

Time taken by particle performing uniform circular motion (U.C.M) to complete one revolution is called period of revolution or periodic time or period.

S.I. Unit :- second (sec)

Dimensions :- $[M^0 L^0 T^1]$

Expression for Period :-

We know,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$T = \frac{\text{Circumference of the circle}}{\text{Speed}}$$

$$T = \frac{2\pi r}{V}$$

$$\boxed{V = r\omega}$$

$$T = \frac{2\pi \cancel{r}}{\cancel{r}\omega}$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

Frequency :-

The number of revolutions performed by a particle performing uniform circular motion in unit time is called as frequency.

It is denoted by f or n

SI Unit :- Hz (s^{-1})

Dimensions :-

$$[f] = [M^0 L^0 T^{-1}]$$

Expressions for frequency :-

$$\text{frequency} = \frac{1}{\text{Time period}}$$

$$f = \frac{1}{T}$$

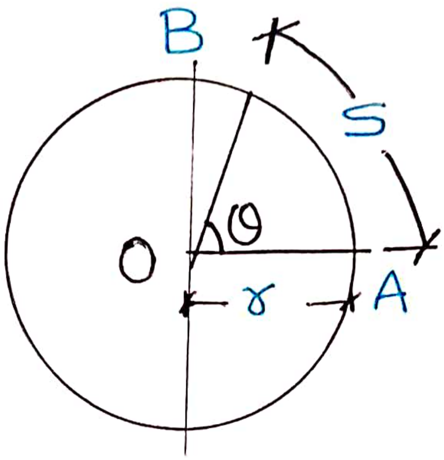
$$\boxed{T = \frac{2\pi}{\omega}}$$

$$f = \frac{1}{\frac{2\pi}{\omega}}$$

$$f = \frac{\omega}{2\pi}$$

$$\boxed{\omega = 2\pi f}$$

Relation between Angular & linear speed or velocity



We know,

$$\text{Angle} = \frac{\text{Arc}}{\text{Radius}}$$

$$\theta = \frac{s}{r}$$

$$\boxed{s = \theta \times r}$$

Differentiate with respect to time,

$$\frac{ds}{dt} = \frac{d}{dt}(\theta \times r)$$

$$\frac{ds}{dt} = \frac{d\theta}{dt} \times r$$

$$\boxed{V = \omega \times r} \qquad \left[V = \frac{ds}{dt} , \ \omega = \frac{d\theta}{dt}\right]$$

In vector form,

$$\boxed{\vec{V} = \vec{\omega} \times \vec{r}}$$

- Obtain the relation between the magnitude of linear acceleration & angular acceleration in circular motion.

Linear Acceleration :-

The rate of change of linear velocity with respect to time is called Linear acceleration ($\vec{a}$)

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\boxed{a = \frac{dv}{dt}} \quad \text{--- (1)}$$

Magnitude of Linear acceleration

We know,

$$\boxed{v = r\omega}$$

Where,

r = radius of circular path

ω = Angular velocity

Put the value $\boxed{v = r\omega}$ in eqn (1),

$$a = \frac{d}{dt}(r\omega)$$

$$a = r\frac{d\omega}{dt} \quad \text{......} \quad [r = \text{constant}]$$

$$\boxed{a = r\alpha} \quad \text{.....} \quad \left[\frac{d\omega}{dt} = \alpha\right]$$

$\cancel{a = r\omega}$ $\boxed{a = r\alpha}$

Linear Acceleration = radius × Angular Acceleration
(a) (r) (α)

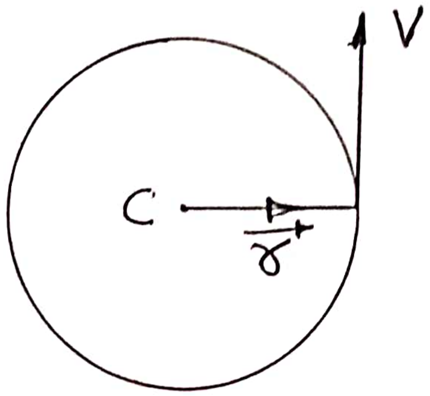
In vector Form,

$$\boxed{\vec{a} = \vec{\alpha} \times \vec{r}}$$

The linear acceleration is in the tangential direction, therefore it is also called as <u>tangential linear acceleration</u>

where,
a_t = Tangential linear acceleration.

Relation between linear & Angular acceleration,



$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$\vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\boxed{\vec{a} = (\vec{\omega} \times \vec{V}) + (\vec{\alpha} \times \vec{r})}$$

$$\boxed{\vec{a} = \vec{a}_c + \vec{a}_T}$$

Where,

a_c = Centripetal acceleration

a_T = Tangential acceleration.

Uniform Circular Motion (Radial Acceleration)

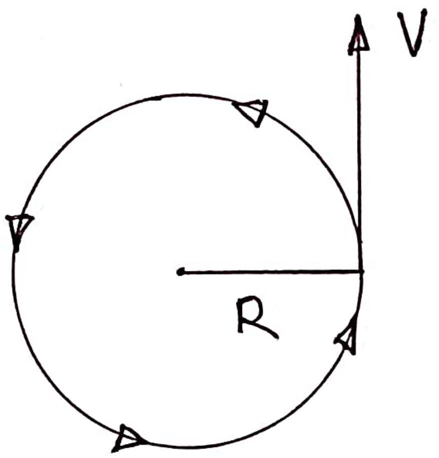
Expression for Centripetal Acceleration

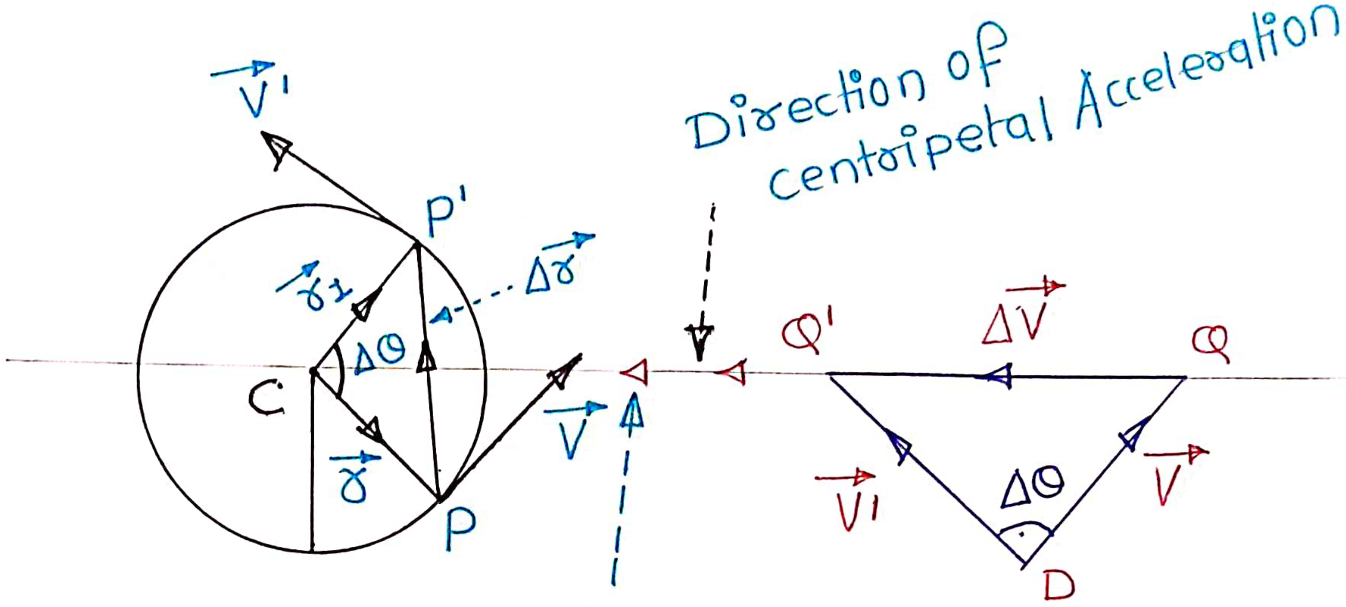
Uniform Circular motion :-

When a particle moves in a circle at a constant speed then the motion is known as uniform circular motion.

Centripetal acceleration or Radial acceleration

The acceleration of a particle performing Uniform circular motion (U.C.M.) which is directed towards the centre & along the radius of circular path is known as centripetal or radial acceleration.





a_c = Centripetal Acceleration

"Acceleration always in change in velocity direction."

Here,

$$\boxed{\Delta\vec{r} = \vec{r}' - \vec{r}}$$

Also,

$$\boxed{\Delta\vec{V} = \vec{V}' - \vec{V}}$$

$$|\vec{r}| = |\vec{r}'| = R$$

$$|\vec{V}| = |\vec{V}'| = V$$

Here,

$\Delta PCP'$ & $\Delta QDQ'$ are similar triangle,

$$\frac{CP}{PP'} = \frac{DQ}{QQ'}$$

$$\frac{|\vec{r}|}{|\Delta \vec{r}|} = \frac{|\vec{V}|}{|\Delta \vec{V}|}$$

$$\frac{R}{\Delta r} = \frac{V}{\Delta V}$$

$$\Delta V = \Delta r \left(\frac{V}{R}\right)$$

Divided by Δt,

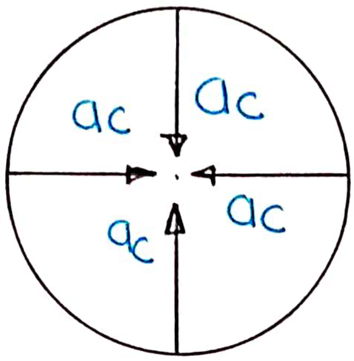
$$\frac{\Delta V}{\Delta t} = \frac{\Delta r}{\Delta t}\left(\frac{V}{R}\right)$$ ---- $\left[a = \frac{\Delta V}{\Delta t}, \; V = \frac{\Delta r}{\Delta t}\right]$

$$a = v\left(\frac{V}{R}\right)$$

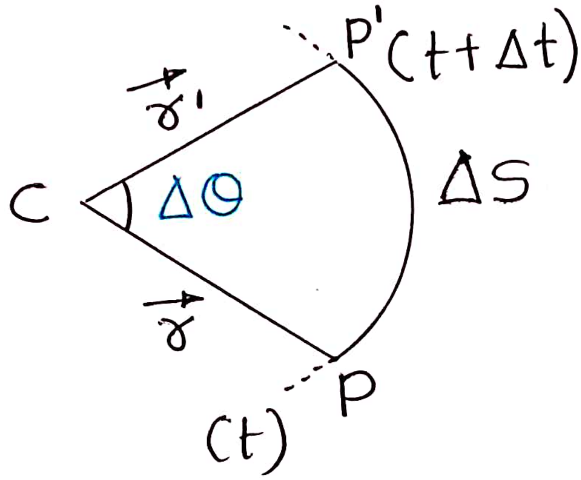
$$\boxed{a = \frac{V^2}{R}}$$

$$\boxed{a_c = \frac{V^2}{R}}$$ ----- This is centripetal Acceleration

Important Results



"Centripetal acceleration is not constant vector."



Where,

$\Delta\theta$ = Angular distance

$\omega = \dfrac{\Delta\theta}{\Delta t}$ --- Angular speed

$\text{Angle} = \dfrac{\text{Arc}}{\text{Radius}}$

Arc = Radius × Angle

$PP' = R \times \Delta\theta$

$\Delta S = R\Delta\theta$

Divided by Δt,

$$\frac{\Delta s}{\Delta t} = R\left(\frac{\Delta \theta}{\Delta t}\right)$$

$$\boxed{V = R\omega}$$

Now,

centripetal acceleration is,

$$a_c = \frac{V^2}{R} \quad \text{.......} (i)$$

$\boxed{V = R\omega}$ put in eqn (i),

$$a_c = \frac{(R\omega)^2}{R}$$

$$a_c = \frac{R^{\cancel{2}}\omega^2}{\cancel{R}}$$

$$a_c = R\omega^2$$

$$\boxed{a_c = \omega^2 R}$$

So, speed

$$\boxed{V = \frac{2\pi R}{T}} \quad \text{------ (ii)}$$

$$\boxed{f = \frac{1}{T}}$$ Put in equation (2)

$$V = 2\pi R \times \frac{1}{T}$$

$$\boxed{V = 2\pi R f}$$

centripetal acceleration is,

$$a_c = \frac{V^2}{R}$$

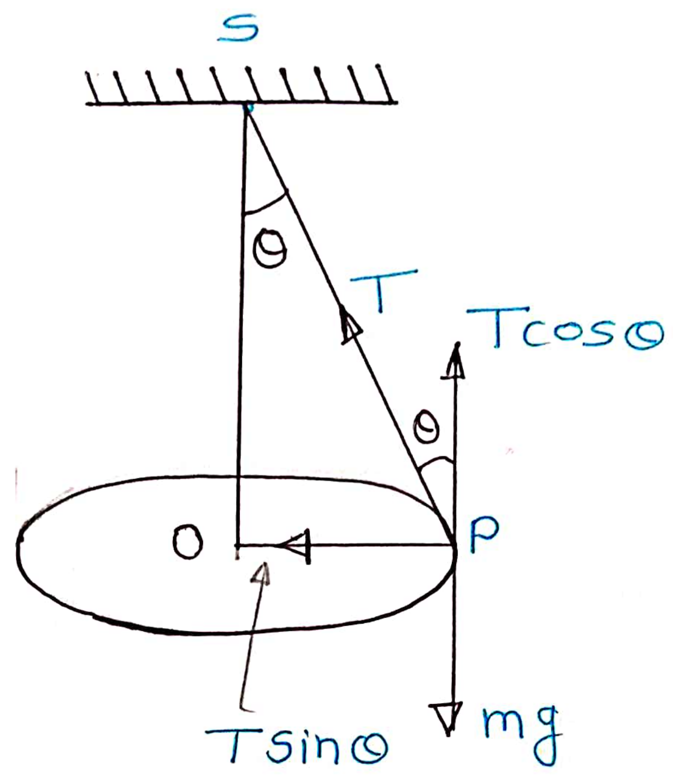
$$\boxed{V = 2\pi R f}$$

$$a_c = \frac{(2\pi R f)^2}{R}$$

$$a_c = \frac{4\pi^2 R^{\cancel{2}} f^2}{\cancel{R}}$$

$$\boxed{a_c = 4\pi^2 R f^2}$$

Conical Pendulum



Where,

S = point of suspension

m = mass of the bob

mg = weight of the bob

l = length of pendulum or string

h = Axial height of cone

r = radius of horizontal circle

T = Tension in the string

Forces acting on the bob are,

1) Tension in the string $= T$

2) Weight of bob $= mg$

$$T\sin\theta = \frac{mv^2}{r} \quad \text{......(1)}$$

$$T\cos\theta = mg \quad \text{-----(2)}$$

From fig,

$$\sin\theta = \frac{r}{L}$$

$$\boxed{r = L\sin\theta}$$

Dividing equation 1 & 2,

$$\frac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

$$\tan\theta = \frac{\frac{mv^2}{r}}{mg}$$

$$\tan\theta = \frac{\cancel{m}v^2}{r\cancel{m}g}$$

$$\tan\theta = \frac{v^2}{rg}$$

$$v^2 = rg\tan\theta$$

$$\boxed{v = \sqrt{rg\tan\theta}}$$

For time period :-

$$T = \frac{2\pi r}{v} \quad \text{------}(3)$$

$v = \sqrt{rg\tan\theta}$ Put in eqn (3),

$$T = \frac{2\pi r}{\sqrt{rg\tan\theta}}$$

$$T = 2\pi\sqrt{\frac{r^{\cancel{2}}}{\cancel{r}g\tan\theta}}$$

$$\boxed{T = 2\pi\sqrt{\frac{r}{g\tan\theta}}} \quad \text{----}(4)$$

$$\sin\theta = \frac{r}{L}$$

$$\boxed{r = L\sin\theta}$$

Putting $\boxed{r = L\sin\theta}$ in eq^n (4),

$$T = 2\pi\sqrt{\frac{L\sin\theta}{g\tan\theta}}$$

$$T = 2\pi\sqrt{\frac{L\cancel{\sin\theta}}{g\frac{\cancel{\sin\theta}}{\cos\theta}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L\cos\theta}{g}}} \quad \text{------(5)}$$

- This is the expression for period of conical pendulum in terms of length.

From ΔSOP,

$$\cos\theta = \frac{h}{L}$$

$$\boxed{h = L\cos\theta}$$

Putting this value in equation (5),

$$\boxed{T = 2\pi\sqrt{\frac{h}{g}}}$$

This is the expression for conical pendulum in terms of height.

Banking of road

Types of curved road,

(1) Horizontal curved road.
(Plane curved road)

(2) Banked curved road.

(1) Horizontal curved road :-

A road whose two edges are horizontal (at same level) is called horizontal curved road, also called curved level road.

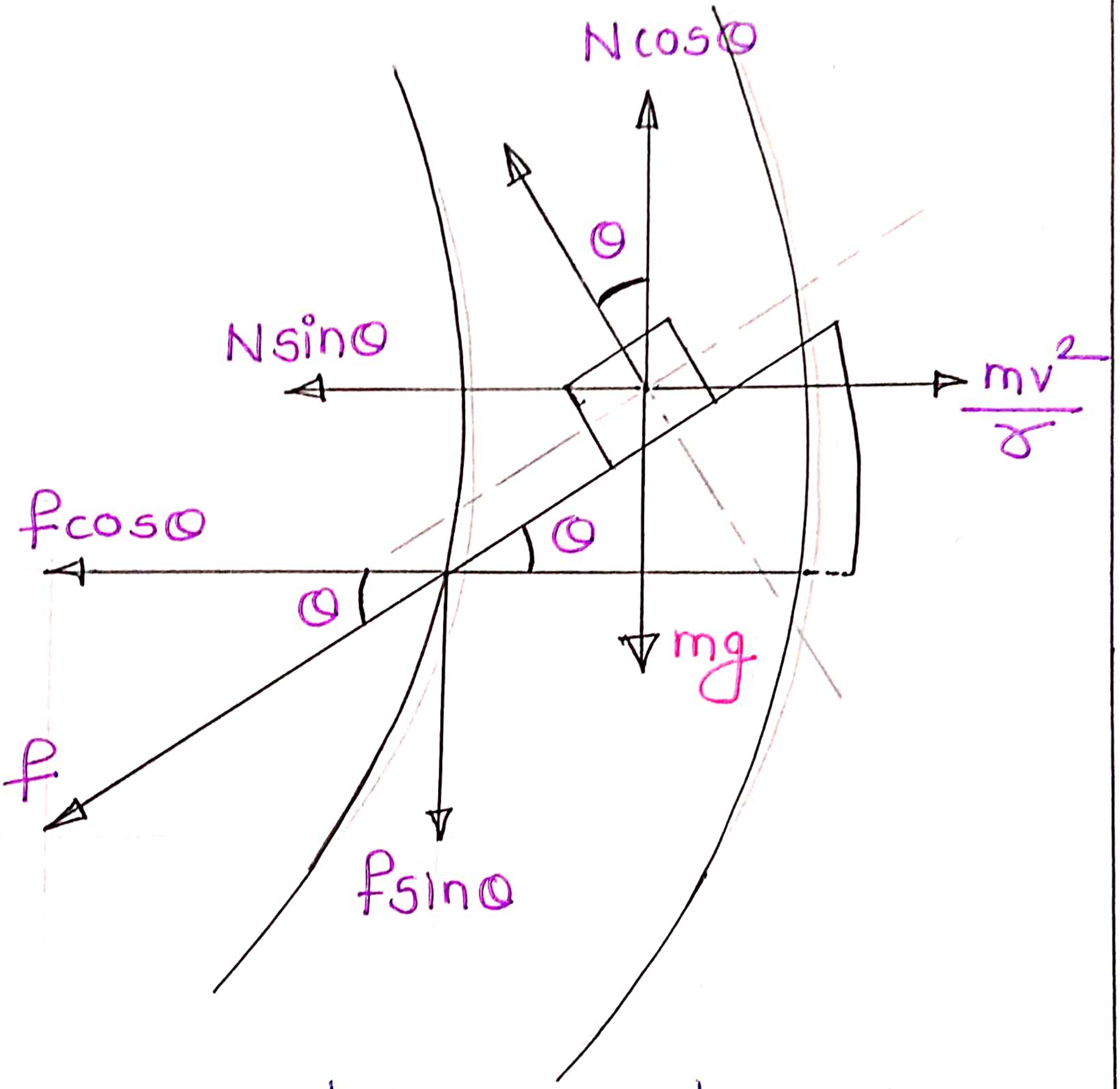
(2) Banked curved road :- (Banking of road)

The road surface is kept inclined to horizontal so that outer edge of road is at higher level than inner edge.

This construction of road is called as banking of road.

Angle of Banking :- (θ)

The angle made by banked road surface with horizontal is called as the angle of banking.



For horizontal component,

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \text{---- (1)}$$

For vertical component,

$$N\cos\theta - f\sin\theta = mg \quad \text{---- (2)}$$

Dividing eqn (1) & (2),

$$\frac{N\sin\theta + f\cos\theta = \frac{\cancel{m}v^2}{r}}{N\cos\theta - f\sin\theta = \cancel{m}g}$$

$$\frac{V^2}{rg} = \frac{N\sin\theta + f\cos\theta}{N\cos\theta - f\sin\theta}$$

Divide R.H.S. by $N\cos\theta$,

$$\frac{V^2}{rg} = \frac{N\sin\theta + f\cos\theta}{N\cos\theta - f\sin\theta}$$

$$\frac{V^2}{rg} = \frac{\frac{\cancel{N}\sin\theta}{\cancel{N}\cos\theta} + \frac{f\cos\theta}{N\cos\theta}}{\cancel{\frac{N\cos\theta}{N\cos\theta}} - \frac{f\sin\theta}{N\cos\theta}}$$

$$\frac{V^2}{rg} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{f\cancel{\cos\theta}}{N\cancel{\cos\theta}}}{1 - \frac{f\sin\theta}{N\cos\theta}}$$

$$\frac{V^2}{rg} = \frac{\tan\theta + \frac{f}{N}}{1 - \frac{f}{N}\tan\theta} \quad \ldots \left[\frac{\sin\theta}{\cos\theta} = \tan\theta\right]$$

$$\frac{V^2}{rg} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta} \quad \cdots\cdots \left[\frac{f}{N} = \mu\right]$$

$$V^2 = \left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right) rg$$

$$\boxed{V = \sqrt{rg\left(\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right)}}$$

This is the expression for maximum speed to avoid the slipping of vehicle.

Important Result

If road is unbanked $\boxed{\theta = 0^\circ}$

$\tan 0 = 0$

$$\boxed{V_{max} = \sqrt{\mu r g}}$$

Vertical circular motion

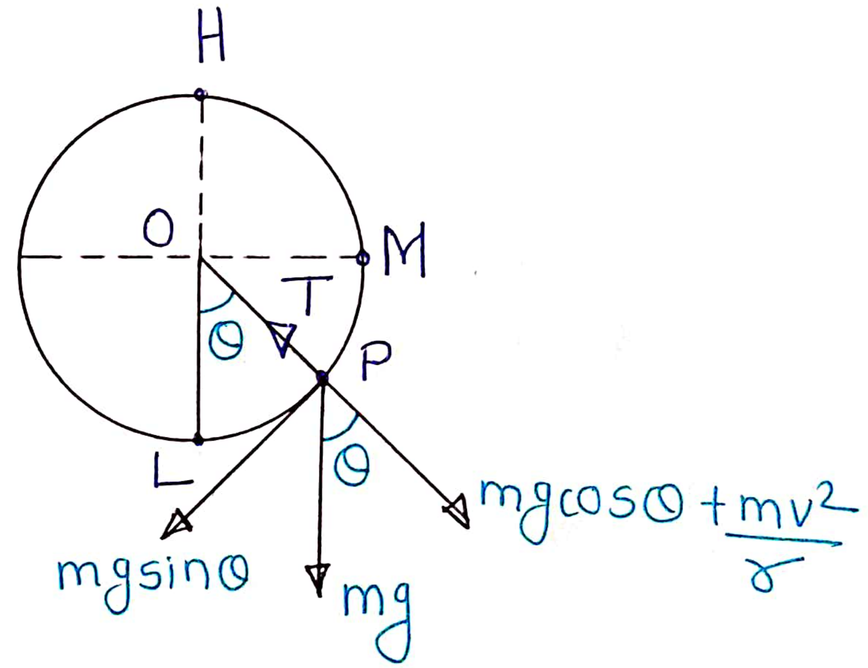
The circular motion of a particle in a vertical plane is called as vertical circular motion.

Examples :-

- Giant wheel

Expression for vertical circular motion,

Diagram :-



Where,

H = Highest position (top position)

L = Lowest position (Bottom position)

M = Horizontal position

T_H = Tension at highest position

T_L = Tension at lowest position

T_M = Tension at Horizontal position.

mg = weight of the body.

Forces acting on the body at position P are,

i] Tension T in the string

ii] weight mg of the body acting vertically downwards.

$$\boxed{T = mg\cos\theta + \frac{mv^2}{r}} \quad \text{----(1)}$$

This is the expression of tension at any position.

1) Highest position (H)

$$\boxed{\theta = 180^\circ}$$

$$\cos 180^\circ = -1$$

Equation (1) becomes,

$$T_H = mg\cos(180^\circ) + \frac{mv^2}{r}$$

$$T_H = -mg + \frac{mv^2}{r}$$

$$\boxed{T_H = \frac{mv^2}{r} - mg} \quad \text{----(ii)}$$

Thus tension in the string is minimum at highest position.

2) Lowest position (L)

$$\boxed{\theta = 0^\circ}$$

$\cos 0 = 1$

Equation (1) becomes,

$$T_L = mg\cos\theta + \frac{mv^2}{r}$$

$\cos 0^\circ = 1$

$$T_L = mg\cos 0 + \frac{mv^2}{r}$$

$$\boxed{T_L = mg + \frac{mv^2}{r}}$$

Thus, tension at lowest position is maximum.

3) Horizontal position (M)

$$\boxed{\theta = 90^\circ}$$

$\cos 90 = 0$

Equation (1) becomes,

$$T_M = mg\cos 90 + \frac{mv^2}{r}$$

$$\boxed{T_M = \frac{mv^2}{r}} \quad \ldots\ldots (\cos 90 = 0)$$

Expression for velocity

1) velocity at highest position (H)

$$T_H = \frac{mv_H^2}{r} - mg \quad \text{----} (1)$$

$$\boxed{T_H = 0}$$

Eqⁿ (1) becomes,

$$0 = \frac{mv_H^2}{r} - mg$$

$$\cancel{m}g = \frac{\cancel{m}v_H^2}{r}$$

$$g = \frac{v_H^2}{r}$$

$$v_H^2 = rg$$

$$\boxed{v_H = \sqrt{rg}} \quad \text{-----} (2)$$

Velocity of highest point is minimum
This speed is called as critical velocity

2) Velocity at lowest position (L)

According to law of conservation of energy,

$$(T.E.)_L = (T.E.)_H$$

$$(K.E.)_L + (P.E.)_L = (K.E.)_H + (P.E.)_H$$

$$\frac{1}{2}mv_L^2 + 0 = \frac{1}{2}mv_H^2 + mg(2r)$$

$$\frac{1}{2}mv_L^2 = \frac{1}{2}mv_H^2 + \frac{1}{2}(4mgr)$$

$$\cancel{\frac{1}{2}}\cancel{m}v_L^2 = \cancel{\frac{1}{2}}\cancel{m}(v_H^2 + 4gr)$$

$$\boxed{V_L^2 = V_H^2 + 4gr} \quad \cdots (3)$$

Put $V_H = \sqrt{rg}$ in eqn (3),

$$V_L^2 = (\sqrt{rg})^2 + 4gr$$

$$V_L^2 = rg + 4gr$$

$$V_L^2 = 5gr$$

$$\boxed{V_L = \sqrt{5gr}}$$

Velocity at any point (P)

According to law of conservation of energy,

$$U_i + K_i = U_f + K_f$$

$$mgh + \frac{1}{2}mV_L^2 = mgh + \frac{1}{2}mV_P^2$$

$$0 + \frac{1}{2}mV_L^2 = \frac{1}{2}mV_P^2 + mgh$$

$$\frac{1}{2}mV_L^2 = \frac{1}{2}mV_P^2 + \frac{1}{2}(mgh) \times 2$$

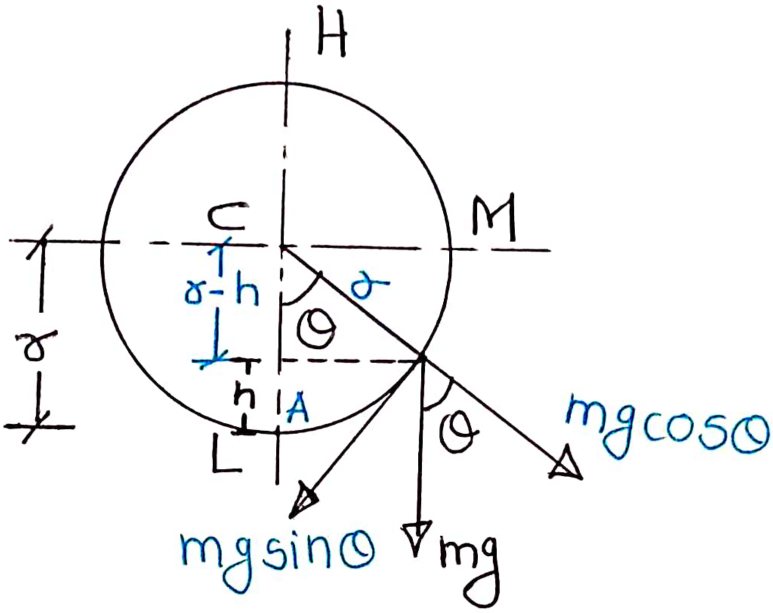
$$\frac{1}{2}mV_L^2 = \frac{1}{2}mV_P^2 + \frac{1}{2}(2mgh)$$

$$\cancel{\frac{1}{2}}\cancel{m}V_L^2 = \cancel{\frac{1}{2}}\cancel{m}(V_P^2 + 2gh)$$

$$V_L^2 = V_P^2 + 2gh$$

$$V_P^2 = V_L^2 - 2gh$$

$$\boxed{V_P = \sqrt{V_L^2 - 2gh}} \quad \text{-----} (1)$$



In ΔACP,

$$\cos\theta = \frac{r-h}{r}$$

$$r-h = r\cos\theta$$

$$r - r\cos\theta = h$$

$$\boxed{h = r - r\cos\theta}$$

Putting $\boxed{h = r - r\cos\theta}$ in eqn (1),

$$V_P = \sqrt{V_L^2 - 2g(r - r\cos\theta)}$$

$V_L = \sqrt{5gr}$ Put in above eqn,

$$V_P = \sqrt{(\sqrt{5gr})^2 - 2g(r - r\cos\theta)}$$

$$V_P = \sqrt{5gr - 2g(r - r\cos\theta)}$$

$$V_p = \sqrt{5gr - 2gr(1-\cos\theta)}$$

$$V_p = \sqrt{gr[5-2(1-\cos\theta)]}$$

$$V_p = \sqrt{gr[5-2+2\cos\theta]}$$

$$\boxed{V_p = \sqrt{gr(3+2\cos\theta)}}$$

- This is the expression for velocity at any position.