Rigid Body :-

A rigid body is defined as, the body in which the distance between any two particles does not change under the action of any force.

Rotational motion :-

When a rigid body moves in such a way that all the particles of body move along circles, the motion of the body is called rotational motion.

Types of Motion,

Types of Motion

1. Linear Motion (Translatory)
2. Pure rotational motion
3. combined translation & rotational

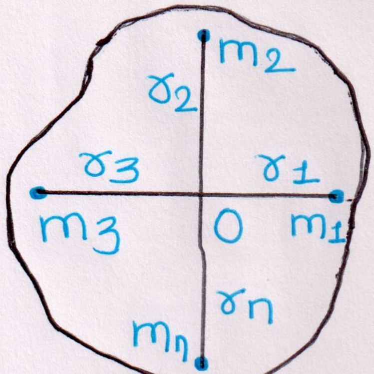
Axis of rotation :-

The imaginary line passing through the body about which the different particles of the body will revolve in different circles, is called the axis of rotation.

Moment of Inertia (I)

M.I. of a rigid body about a given axis of rotation is defined as the sum of the products of mass of each particle of the body & square of its distance from the axis of rotation.

Diagram :-



Moment of inertia I about the given axis is given as,

$$\boxed{I = m_1 r_1^2 + m_2 r_2^2 + \cdots\cdots + m_n r_n^2}$$

$$\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$$

Unit :- kg/m^2 (S.I.)

CGS :- gm/cm^2

Dimensions :-

$$[I] = [M^1 L^2 T^0]$$

Radius of Gyration :- (k)

The radius of gyration of a body about a given axis of rotation is defined as, the distance between the axis of rotation & the point at which the whole mass of the body can be supposed to be concentrated so as to have the same M. I. as that of the body about the given axis.

where,

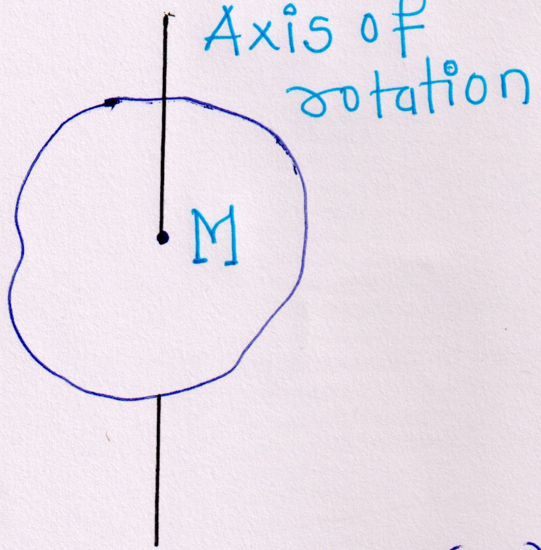
M = mass of the body

K = Radius of gyration

I = moment of inertia

$$I = Mk^2$$

$$\boxed{k = \sqrt{\frac{I}{M}}}$$



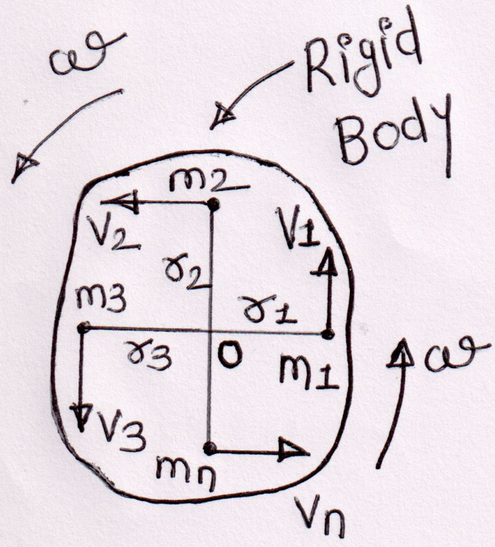
S.I unit :- metre (m)

CGS :- cm

Dimensions :- $[k] = [M^0 L^1 T^0]$

K.E (kinetic Energy) of a rotating body

(Uniform speed)



- consider a rigid body rotating about an axis passing through a point O with constant angular velocity. (ω)
- A body consist of n number of particles having masses $m_1, m_2, \dots m_n$ situated at distances $r_1, r_2, \dots r_n$ from the axis of rotation.
- linear velocities be $V_1, V_2, V_3, \dots V_n$.

Linear velocities of first particle is,

$$\boxed{V_1 = r_1 \omega}$$

K.E. of first particle is,

$$E_1 = \frac{1}{2} m_1 V_1^2$$

$$\boxed{V_1 = r_1 \omega}$$

$$E_1 = \frac{1}{2} m_1 (r_1 \omega)^2$$

$$\boxed{E_1 = \frac{1}{2} m_1 r_1^2 \omega^2}$$

K.E. of second particle is,

$$E_2 = \frac{1}{2} m_2 v_2^2$$

$$\boxed{v_2 = r_2 \omega} \quad \text{---} [v = r\omega]$$

$$E_2 = \frac{1}{2} m_2 (r_2 \omega)^2$$

$$\boxed{E_2 = \frac{1}{2} m_2 r_2^2 \omega^2}$$

Similarly, K.E. of nth particle,

$$\boxed{E_n = \frac{1}{2} m_n r_n^2 \omega^2}$$

Now,

The total K.E. of the body is,

$$E = E_1 + E_2 + E_3 + \text{-----} + E_n$$

$$E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \text{------} + \frac{1}{2} m_n r_n^2 \omega^2$$

$$E = \frac{1}{2} \omega^2 \left[m_1 r_1^2 + m_2 r_2^2 + \text{------} + m_n r_n \right]$$

$$E = \frac{1}{2} \omega^2 \left[\sum_{i=1}^{n} m_i r_i^2 \right]$$

But,

$$\boxed{\sum_{i=1}^{n} m_i r_i^2 = I}$$

$$\boxed{E = \frac{1}{2} I\omega^2}$$

This is the expression for K.E. of uniformly rotating body.

Important Results

1] $E = \frac{1}{2} I\omega^2$

$E = \frac{1}{2} I\omega \times \omega$

[$I\omega = L$ - Angular momentum]

$$\boxed{E = \frac{1}{2} L\omega}$$

2] $E = \frac{1}{2} I\omega^2$

multiply & divide by 'I'

$E = \frac{1}{2} \frac{I \times I}{I} \omega^2$

$E = \frac{1}{2} \frac{I^2 \omega^2}{I}$

$$L = I\omega \ \& \ I = Mk^2$$

$$E = \frac{1}{2}\frac{L^2}{Mk^2}$$

$$\boxed{E = \frac{1}{2M}\left(\frac{L}{k}\right)^2}$$

3] $\boxed{\omega = 2\pi n}$

$$E = \frac{1}{2} I\omega^2$$

$$E = \frac{1}{2} I (2\pi n)^2$$

$$E = \frac{1}{2} I (2)^2 \pi^2 n^2$$

$$E = \frac{1}{\cancel{2}} \times \overset{2}{\cancel{4}}\, I \pi^2 n^2$$

$$\boxed{E = (2 I \pi^2) n^2}$$

$$E = (\text{constant})\, n^2$$

$$\boxed{E \propto n^2}$$

K.E of a rotating body is directly proportional to the square of the frequency of the body.

Torque :-

or

Moment of Force

Torque or moment of force about a point is defined as the product of the force & the perpendicular distance of the point of application of the force from the point.

It is denoted by 'τ'

S.I. Unit :- N-m or J

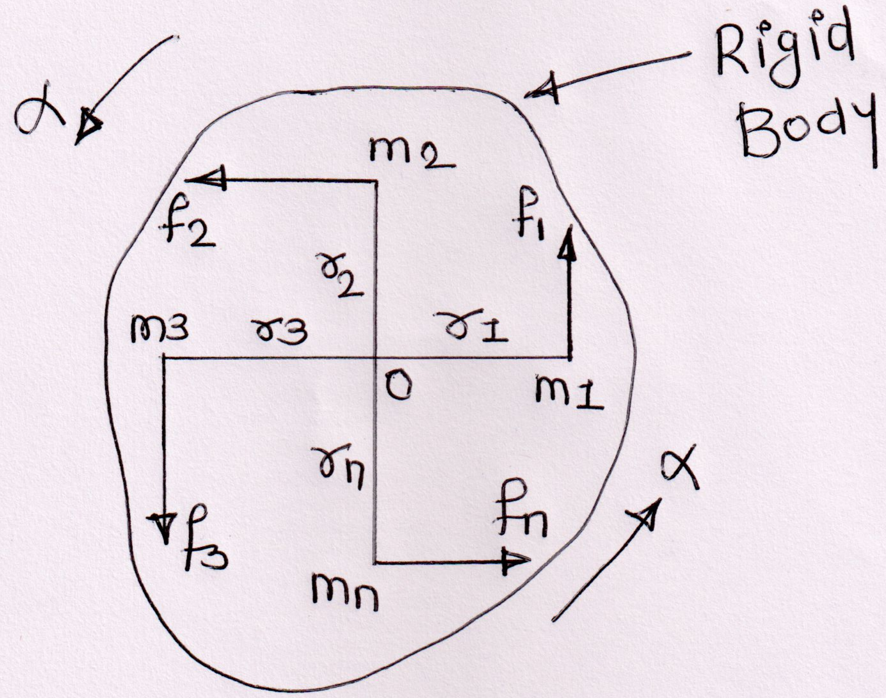
CGS unit :- dyne-cm or erg

Dimensions :-

$$[\tau] = [M^1 L^2 T^{-2}]$$

Torque

Relation between torque (τ) & angular acceleration (α)



Where,

$m_1, m_2, m_3 ---- m_n$ – particles masses

$r_1, r_2, r_3 --- r_n$ – distances from the axis of rotation.

Linear acceleration of first particle is,

$$\boxed{a_1 = r_1 \alpha}$$

Force acting on first particle is,

$$f_1 = m_1 a_1$$

$$\boxed{a_1 = r_1 \alpha}$$

$$\boxed{f_1 = m_1 r_1 \alpha} \quad \text{----} \left[a_1 = r_1 \alpha\right]$$

Torque acting on the first particle is,

Torque = force x moment arm

$$\tau_1 = f_1 \times r_1$$

$$\tau_1 = f_1 \times r_1$$

$$\boxed{\tau_1 = m_1 r_1 \alpha r_1} \quad \boxed{\tau_1 = m_1 r_1^2 \alpha}$$

Torque acting on the second particle is,

$$\boxed{\tau_2 = m_2 r_2^2 \alpha}$$

Torque acting on the nth particle is,

$$\boxed{\tau_n = m_n r_n^2 \alpha}$$

The total torque acting on the body is,

$$\tau = \tau_1 + \tau_2 + \tau_3 + \cdots + \tau_n$$

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \cdots + m_n r_n^2 \alpha$$

$$\tau = \alpha \left[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2\right]$$

$$\tau = \left[\sum_{i=1}^{n} m_i r_i^2\right] \alpha$$

But,

$$\sum_{i=1}^{n} m_i r_i^2 = I$$

$$\boxed{\tau = I\alpha}$$

This is an expression for torque acting on the body.

<u>S.I. unit</u> :- <u>N-m</u> or J

<u>CGS unit</u> :- dyne-cm or erg

<u>Dimensions</u> :- $[\tau] = [M^1 L^2 T^{-2}]$

Angular Momentum :- (L)

Defn :-

Angular momentum of rotating body about a given axis, is defined as the product of its moment of inertia about the given axis & its angular velocity.

$$\boxed{L = I\omega}$$

Let,

L = Angular momentum

I = Moment of inertia

ω = Angular velocity

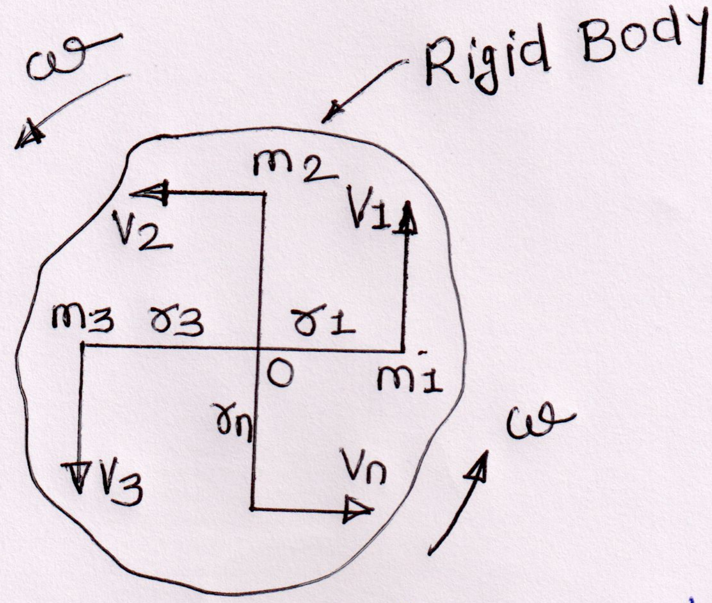
S.I. unit = $kg\frac{m^2}{s}$

CGS unit = $gm\frac{cm^2}{s}$

Dimensions

$$[L] = [M^1 L^2 T^{-1}]$$

Expression for angular momentum



- consider a rigid body rotating about an axis passing through point O,
- m_1, m_2, m_3 having particles masses
- r_1, r_2, r_3 distance from axis of rotation.
- V_1, V_2, V_3 linear velocities.

The linear velocities of the particle is,

$$\boxed{V_1 = r_1 \omega}$$

Linear momentum $(P_1) = m_1 V_1$

$$\boxed{P_1 = m r_1 \omega}$$

Angular momentum (L)

$$L_1 = P_1 r_1$$

$$L_1 = m_1 r_1 \omega r_1$$

$$\boxed{L_1 = m_1 r_1^2 \omega}$$

Angular momentum of second particle is,

$$\boxed{L_2 = m_2 r_2^2 \omega}$$

Angular momentum of nth particle is,

$$L_n = m_n r_n^2 \omega$$

The total angular momentum is,

$$L = L_1 + L_2 + L_3 + \cdots + L_n$$

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \cdots + m_n r_n^2 \omega$$

$$L = \omega \left[m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2 \right]$$

$$L = \left[\sum_{i=1}^{n} m_i r_i^2 \right] \omega$$

But,

$$\sum_{i=1}^{n} m_i r_i^2 = I$$

$\boxed{L = I\omega}$ .... This is an angular momentum expression.

Principle of Parallel Axis

Statement:-

The moment of inertia of a body about an axis is equal to the sum of

i) The moment of inertia of the body about a parallel axis passing through its centre of mass &

ii) The product of the mass of the body & square of the distance betn the two axes.

$$\boxed{I_o = I_c + Mh^2}$$

Where,

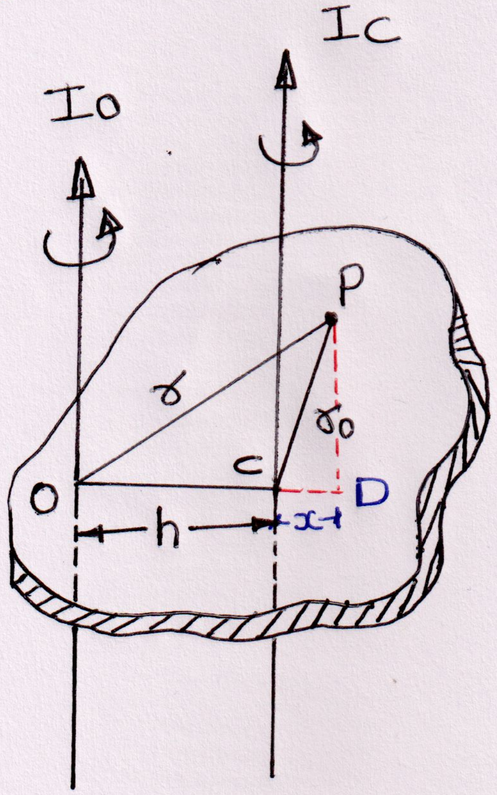
I_o = Moment of inertia (M.I) of the body about any axis passing through point O.

I_c = Moment Inertia of the body about a parallel axis passing through its centre of mass (c).

M = Mass of the body

h = distance between the two parallel axes.

Diagram :-



$$I_o = \int r^2 \, dm$$

$$I_c = \int r_0^2 \, dm$$

Apply pythagoras theorem,

In ΔOPD,

$$OP^2 = PD^2 + OD^2$$

$$OP^2 = PD^2 + (OC + CD)^2$$

$$OP^2 = PD^2 + (h+x)^2$$

$$\boxed{r^2 = PD^2 + h^2 + 2hx + x^2} \quad \text{---(1)}$$

Now,

In ΔCDP,

Apply pythagoous theorm,

$$CP^2 = CD^2 + DP^2$$

$$CP^2 = CD^2 + PD^2$$

$$r_0^2 = x^2 + PD^2$$

$$\boxed{r_0^2 = PD^2 + x^2}$$

Put $r_0^2 = PD^2 + x^2$ in eqn (1),

$$\boxed{r^2 = r_0^2 + h^2 + 2hx}$$

Multiplying both sides by $\underline{dm}$ & integrating,

$$\int r^2 dm = \int r_0^2 dm + \int h^2 dm + \int 2hx\, dm \quad \text{---(2)}$$

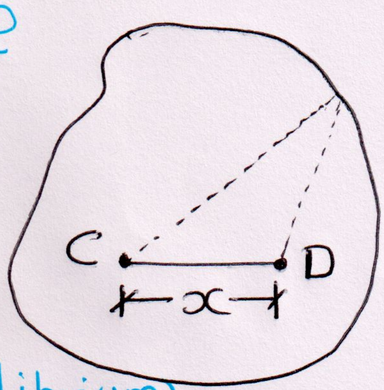
$I_o = \int r^2 dm$,

$I_c = \int r_0^2 dm$, Put in (2),

$$I_0 = I_c + h^2 M + 0$$

$$\int x \, dm = 0$$

As algebric sum of all moments of all particles about C.O.M. is always zero (when body is in equilibrium)



$$\boxed{I_0 = I_c + Mh^2}$$

Thus, principle of parallel axes is approved.

Principle of Perpendicular Axes

Statement :-

The M.I. of a plane lamina about an axis perpendicular to its plane is equal to the sum of its M.I. about two mutually perpendicular axes in the plane of lamina & intersecting at the point where the perpendicular axis cuts the lamina.

$$\boxed{I_z = I_x + I_y}$$

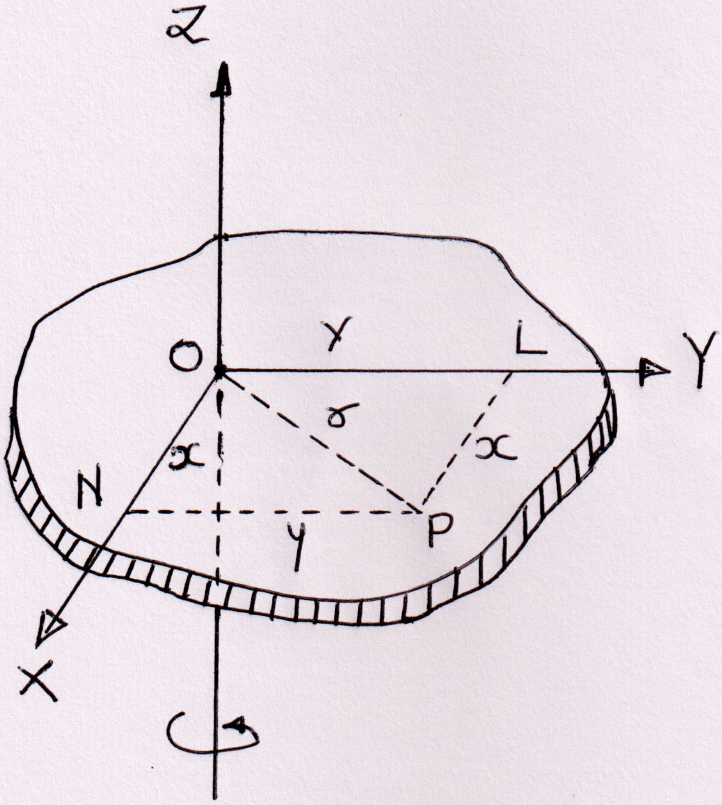
Where,

I_z = M.I. of lamina about Z-axis

I_x = M.I of lamina about x-axis

I_y = M.I. of lamina about Y-axis

Perpendicular Axis theorem,



$$I_x = \int y^2 \, dm$$

$$I_y = \int x^2 \, dm$$

$$I_z = \int r^2 \, dm$$

In ΔOLP,

Apply pythagoras theorem,

$$OP^2 = LP^2 + OL^2$$

$$r^2 = x^2 + y^2$$

Multiplying by 'dm' on both sides & Integrating,

$$\int r^2 dm = \int x^2 dm + \int y^2 dm$$

$$\boxed{I_z = I_y + I_x}$$

$$\cdots\left[I_x = \int y^2 dm,\ I_y = \int x^2 dm \quad I_z = \int r^2 dm\right]$$

Thus, principle of perpendicular axes is proved.